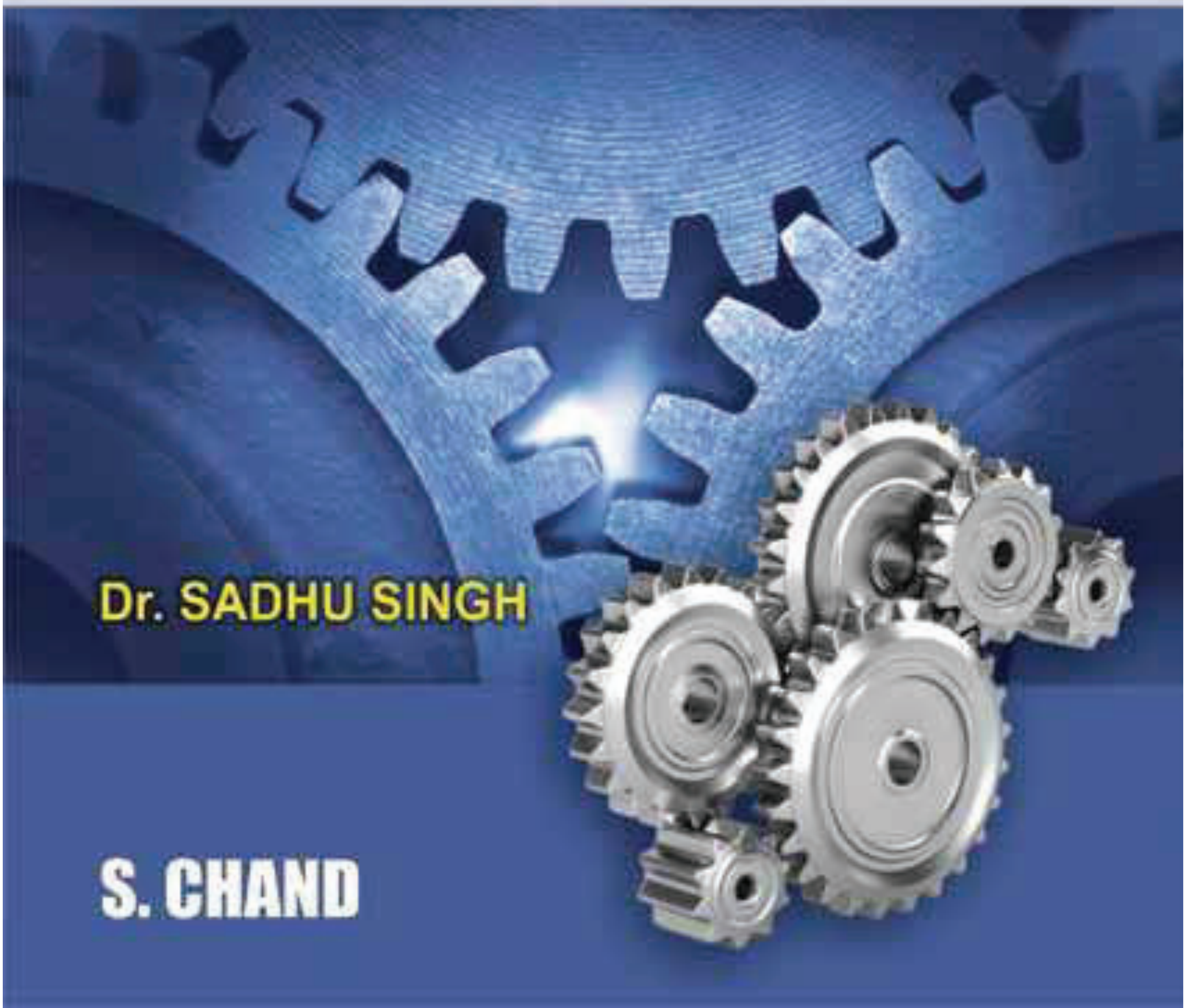
2ND EDITION

Handbook of Mechanical Engineering



Dr. SADHU SINGH

S. CHAND

Handbook of Mechanical Engineering

HANDBOOK OF
MECHANICAL ENGINEERING

For the Students of B.E./B.Tech. (Mechanical Engineering). Also Useful for Engineering Services / Civil Services / Forest Services / GATE / State Services and Other Competitive Examinations

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Dedicated to
the
sweet memory
of my wife
Smt. Manjit Kaur
who passed away
on 20th December, 2009
after a great struggle with life

PREFACE TO THE SECOND EDITION

I am pleased to present the Second Edition of the book. This edition is revised keeping in mind the requirements of the students. In this edition, several solved examples and questions have been incorporated to reinforce the students' understanding of the subject matter.

I hope this edition would be more useful to the students as well as to the aspirants of various competitive examinations such as Indian Engineering Services, Civil Services, Forest Services, GATE, State Services, etc.

I am thankful to the management and the editorial team of S.Chand & Company Pvt. Ltd., New Delhi for help and support in publication of this edition.

Any comments and suggestions for the improvement of the book will be gratefully acknowledged.

Dr. Sadhu Singh

PREFACE TO THE FIRST EDITION

It is another attempt to place before the candidates, in the book form, the full contents of the subject of **Mechanical Engineering.** It was a long cherished dream of the author to bring out such a book as per the wishes of his late wife, Smt. Manjit Kaur. She has been the driving force for the last about forty-one years for writing.

The book contains 28 chapters. The subject matter has been divided into five major areas of Engineering Mathematics, Design Engineering, Production Engineering, Industrial Engineering, and Thermal Engineering. Assertion and Reason, Short Answer Type Questions and Glossary of Terms in Mechanical Engineering have been covered in the last three chapters.

The book has been written as per the syllabi of Engineering Services, Civil Services, Forest Services, GATE, State Services, and other Competitive Examinations. It is hoped that the book shall be quite useful to the candidates preparing for these examinations.

The moral support received from Mrs. Narinderpal Kaur (my daughter-in-law) and grandchildren Kanupreet and Amitoj is praiseworthy.

The support and cooperation received from the management and the editorial team of S. Chand & Company Pvt. Ltd., New Delhi is highly acknowledged.

Suggestions for the further improvement of the book are welcome and shall be duly acknowledged.

Dr. Sadhu Singh

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PART – I
ENGINEERING MATHEMATICS

CHAPTER 1

ENGINEERING MATHEMATICS

1.1 LINEAR ALGEBRA

1.1.1 Matrices

A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n ($m \times n$) matrix. Thus

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

a_{ij} are the elements of the matrix.

Special Matrices

Row matrix—has a single row.

Column matrix—has a single column.

Square matrix—has same number of rows and columns.

Diagonal matrix—has all elements zero other than the diagonal elements (a_{ij}, $i = j$).

Unit matrix—is a diagonal matrix having all diagonal elements equal to unity. It is also called identity matrix [I].

Null matrix—whose all elements are zeroes.

Symmetric matrix—is a square matrix such that $a_{ij} = a_{ji}$, $i \neq j$.

Skew-symmetric matrix—is a square matrix such that $a_{ij} = -a_{ji}$, $i \neq j$.

Upper triangular matrix—is a square matrix whose all elements below the leading diagonal are zero.

Lower triangular matrix—is a square matrix whose all elements above the leading diagonal are zero.

Hermitian matrix—A square matrix $A = [a_{ij}]$ in which $(i, j)^{th}$ element is equal to the conjugate complex of the $(j, i)^{th}$ element, *i.e.*, $a_{ij} = \overline{a}_{ji}$ for all i and j.

Skew-Hermitian matrix—A square matrix $A = [a_{ij}]$ in which $a_{ij} = -\overline{a}_{ji}$ for all i and j. Its diagonal elements must be pure imaginary numbers or zero.

Orthogonal matrices—A square matrix A is said to be an orthogonal matrix if $AA^T = A^TA = I$.

Operations on Matrices

1. Addition of matrices.

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$. Then

$$C = [c_{ij}]_{m \times n} = A + B \quad = [a_{ij} + b_{ij}]_{m \times n}$$

$$A + B = B + A$$

2. Subtraction of matrices

$$D = [d_{ij}]_{m \times n} = A - B = [a_{ij} - b_{ij})_{m \times n}$$

$$A - B = -(B - A)$$

3. Multiplication of a matrix by a scalar.

If $A = [a_{ij}]_{m \times n}$ and k is a scalar, then

$$kA = [ka_{ij}]_{m \times n}$$

Also $k(A + B) = kA + kB$

4. Multiplication of matrices.

If $A = [a_{ij}]_{l \times m}$ and $B = [b_{jk}]_{m \times n}$, then

$$C = [c_{ik}]_{l \times n} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

The condition for multiplication is that number of columns in the first matrix should be equal to the number of rows in the second matrix.

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$AB = -BA$$

$$AI = A = IA$$

5. Power of a matrix. If A be a square matrix, then the product AA is defined as A^2. If $A^2 = A$, then the matrix A is called **idempotent**.

Related Matrices

1. **Transpose of a matrix**—is the matrix obtained from any given matrix by interchanging the rows and columns.

If $A = [a_{ij}]$ then $A^T = [a_{ji}]$

$$(A^T)^T = A$$

For a square matrix

$$|A| = |A^T|$$

$$[AB]^T = B^T A^T$$

2. **Adjoint of a square matrix**—is the transposed matrix of cofactors of the given matrix. It is written as *adj A*.

3. **Inverse of a matrix**—If A be any matrix, then a matrix B if it exists, such that $AB = BA = I$, is called the inverse of A. It is denoted by A^{-1}.

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(A^{-1})^k = (A^k)^{-1}$$

$$AA^{-1} = I$$

4. **Rank of a matrix**—A matrix is said to be of rank r when it has at least one non zero minor of order r, and every minor of order higher than r vanishes.
5. Elementary transformation of a matrix.

   The following operations on a matrix are called elementary transformations.

   (*i*) The interchange of any two rows (columns).

   (*ii*) The multiplication of any row (column) by a non-zero number.

   (*iii*) The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).
6. **Equivalent matrix**—Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same rank and order.
7. **Elementary matrices**—An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.

1.1.2 Solutions of Linear System of Equations

1. **Non-homogeneous equations**

$$[a_{ij}]\{x_j\} = \{b_i\}$$

or

$$AX = B$$

where,

$$[a_{ij}] = A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \quad \text{is the coefficient matrix}$$

$$\{x_j\} = X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad \text{is the unknown column matrix}$$

$$\{b_i\} = B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1} \quad \text{is the known column matrix.}$$

(*a*) Cramer's Rule

If $|a_{ij}| \neq 0$, then

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots \\ b_m & a_{m2} & \cdots & a_{mn} \end{vmatrix}}{|a_{ij}|}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots \\ a_{m1} & b_m & \cdots & a_{mn} \end{vmatrix}}{|a_{ij}|}$$

and so on.

(*b*) Matrix method

$$AX = B,\ A^{-1}AX = A^{-1}B \text{ or } IX = A^{-1}B$$

or

$$X = A^{-1}B$$

$$\{x_j\} = \frac{1}{|a_{ij}|}[A_{ij}]\{b_i\}$$

where $[A_{ij}]$ are the cofactors of a_{ij} in the determinant $|a_{ij}|$.

2. Homogeneous equations

$$AX = 0$$

where, $A = [a_{ij}]_{m \times n},\ X = \{x_j\}_{n \times 1},\ O = \{0\}_{m \times 1}$

Let r = rank of matrix A.

If $r = n$, then zero (trivial) solution will be the only solution

If $r < n$, there will be an infinity of solutions.

3. Consistency of Linear system of Non-homogeneous equations.

$$AX = B$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ is the coefficient matrix}$$

Let

$$K = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \text{ be the augmented matrix.}$$

Rouche's theorem for consistency : This theorem states that the system of equations $AX = B$ is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank, otherwise the system is inconsistent.

Procedure to test the consistency :

Let r = rank of matrix A

s = rank of augmented matrix K

(*i*) If $r \neq s$, the equations are inconsistent, *i.e.,* there is no solution.

(*ii*) If $r = s = n$, the equations are consistent and there is a unique solution.

(*iii*) If $r = s < n$, the equations are consistent and there are infinite number of solutions. Giving arbitrary values to $n - r$ of the unknowns, we may express the other a unknowns in terms of these.

4. Consistency of Linear System of Homogeneous Equations.

$$AX = 0$$

Let r = rank of the coefficient matrix A.

(*i*) If $r = n$, the equations $AX = 0$ have only a trivial (zero) solution.

(*ii*) If $r < n$, the equations $AX = 0$ have $(n - r)$ linearly independent solutions.

(*iii*) When $m < n$, *i.e.,* the number of equations is less than the number of variables, the solution is always other than trivial solution. The number of solutions is infinite.

(*iv*) When $m = n$, *i.e.,* the number of equations is equal to the number of variables, the necessary and sufficient condition for non-trivial solutions is that the determinant of the coefficient matrix is zero. In this case, the equations are said to be consistent.

Characteristic Equation

If A is any square matrix of order n, we can form a matrix A – lI, where I is the n^{th} order unit matrix. The determinant of this matrix equation to zero is called the characteristic equation of A. The roots of this equation are called the eigen-values of matrix A. Thus,

$$|A - \lambda I| = \begin{bmatrix} a_{11-\lambda} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22-\lambda} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn-\lambda} \end{bmatrix} = 0$$

or $(-1)^n \lambda^n + k_1 \lambda^{n-1} + ... + k_{n=0}$

where, k's are expressible in terms of the elements a_{ij}.

1.1.3 Eigen Vectors

If λ is a characteristic root or eigen values of A, then a non-zero vector X such that $AX = \lambda X$ is called the eigen vector of A corresponding to the characteristic root λ. Thus,

$$AX = \lambda X$$

or $$AX - \lambda IX = 0$$

or $$[A - \lambda I]X = 0$$

This matrix represents homogeneous linear equations which will have a non-trivial solution only if the coefficient matrix is singular, *i.e.,* if $|A - \lambda I| = 0$, which is the same as the characteristic equation of matrix A.

(*i*) If X is a characteristic vector of matrix A corresponding to the characteristic value λ, then CX is also a characteristic vector of A corresponding to the same characteristic value λ, where C is any non-zero scalar.

(*ii*) Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Properties of Eigen Values

1. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.
2. The product of the eigen values of a matrix A is equal to its determinant.
3. If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}.
4. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its eigen value.
5. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$, where m is a positive integer.

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation, *i.e.,* if the characteristic equation for the n^{th} order square matrix A is

$$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + ... + k_{n=0}$$

then $(-1)^n A^n + k_1 A^{n-1} + ... + k_{n=0}$

Example 1.1 *Find the inverse of the matrix* $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Solution. $$A^{-1} = \frac{adj\ A}{|A|}$$

$$|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1$$

Cofactor matrix is : $\begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 1.2 *Determine the rank of the matrix* $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

Solution. Operating $R_3 - R_1$, $R_4 - R_1$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 0 & 3 & 3 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Operating $C_3 - C_1$, $C_4 - C_1$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Operating $R_3 - 3R_2$, $R_4 - R_2$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_3 + 3C_2$, $C_4 + C_2$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4th order and 3rd order minor of A are zero. Only 2nd order $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$ minor is non-zero.

Hence rank of matrix is 2.

Example 1.3 *Test for the consistency of the following set of equations and solve*

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 9 \\ 5 \end{Bmatrix}$$

Solution. Operate $5R_2 - 3R_1$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 33 \\ 5 \end{Bmatrix}$$

Operate $5R_3 - 7R_1$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 33 \\ -3 \end{Bmatrix}$$

Operate $R_3 + R_2/11$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 33 \\ 0 \end{Bmatrix}$$

Operate $R_2/11$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3 \\ 0 \end{Bmatrix}$$

The rank of the coefficient matrix and augmented matrix is 2.

Hence, the equations are consistent.

$$5x_1 + 3x_2 + 7x_3 = 4$$

$$11x_2 - x_3 = 3$$

$$x_2 = \frac{3}{11} + \frac{x_3}{11}$$

$$5x_1 = 4 - 3\left(\frac{3}{11} + \frac{x_3}{11}\right) - 7x_3$$

$$= 4 - \frac{9}{11} - \frac{3x_3}{11} - 7x_3$$

$$= \frac{35}{11} - \frac{80}{11}x_3$$

$$x_1 = \frac{7}{11} - \frac{16}{11}x_3$$

where, x_3 is a parameter.

$$x_1 = \frac{7}{11},\ x_2 = \frac{3}{11},\ x_3 = 0$$

Example 1.4 *Find the eigen values and the eigen vectors of the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution. The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1 - \lambda)\ [(5 - \lambda)\ (1 - \lambda) - 1] - 1[1 \times (1 - \lambda) - 3] \times 3[1 - 3(5 - \lambda)] = 0$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$\lambda = -2$ satisfies this equation.

$$(\lambda + 2)\ (\lambda^2 - 9\lambda + 18) = 0$$

$$(\lambda + 2)(\lambda - 3)(\lambda - 6) = 0$$
$$\lambda = -2, 3, 6$$

The eigen values of A are –2, 3, 6.

Eigen vectors are $[A - \lambda I]X = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$

Putting $\lambda = -2$, we have

$$3x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 7x_2 + x_3 = 0$$
$$3x_1 + x_2 + 3x_3 = 0$$

The first and third equations being same, we have from the first two equations,

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

or $$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Hence, the eigen vector is (–1, 0, –1)

For $\lambda = 3$

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0$$
$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

The eigen vector is (1, –1, 1)

For $\lambda = 6$

$$-5x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

The eigen vector is (1, 2, 1)

Hence, the three eigen vectors are : (–1, 0, 1), (1, –1, 1), (1, 2, 1).

1.2 DIFFERENTIATION

A function $f(x)$ is said to be differentiable at $x = a$, if both

$$\underset{h \to 0}{\text{Lt}} \frac{f(a+h) - f(a)}{h}, h > 0 \text{ and } \underset{h \to 0}{\text{Lt}} \frac{f(a-h) - f(a)}{-h}, h > 0.$$

exist and have a common value (finite or infinite).

The common value is called the derivative of $f(x)$ at the point $x = a$.

If $y = f(x)$, then its first order and higher order derivatives are written as :

$$f'(x), \frac{dy}{dx}, Dy, f''(x), \frac{d^2y}{dx^2}, D^2y, \text{ and so on.}$$

Some standard results of differentiation are :

$$D^n (ax+b)^m = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}$$

$$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

$$D^n \log (ax+b) = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^x}$$

$$D^n (a^{mx}) = m^n (\log a)^n a^{mx}$$

$$D^n (e^{mx}) = m^n e^{mx}$$

$$D^n \sin (ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

$$D^n \cos (ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

$$D^n [e^{ax} \sin (bx+c)] = (a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

$$D^n [e^{ax} \cos (bx+c)] = (a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

$$D^n (x^n) = n!$$

Leibnitz's Theorem

If u, v be two functions of x possessing derivatives of nth order, then
$(uv)_n = u_n v + {}^nc_1\, u_{n-1}\, v_1 + {}^nc_2\, u_{n-2}\, v_2 + \ldots + {}^nc_r\, u_{n-r}\, v_r + \ldots + {}^nc_n\, uv_n$.

1.3 CALCULUS

1.3.1 Limit

Right hand limit. If x approaches 'a' from the right, *i.e.,* $x > a$, the limit of f is called the right hand limit of $f(x)$, and is written as

$$\underset{x \to a+}{\text{Lt}} f(x) \text{ or } f(x+)$$

Left hand limit. If x approaches 'a' from the left, *i.e.,* $x < a$, the limit of f is called the left hand limit of $f(x)$, and is written as

$$\underset{x \to a-}{\text{Lt}} f(x) \text{ or } f(x-)$$

(*i*) If $f(a+) = f(a-)$, then limit of f as $x \to a$ exists.

(*ii*) $\underset{x \to a}{\text{Lt}} [f_1(x) \pm f_2(x)] = \underset{x \to a}{\text{Lt}} f_1(x) \pm \underset{x \to a}{\text{Lt}} f_2(x)$

(*iii*) $\underset{x \to a}{\text{Lt}} [f_1(x) \cdot f_2(x)] = \underset{x \to a}{\text{Lt}} f_1(x) \cdot \underset{x \to a}{\text{Lt}} f_2(x)$

(*iv*) $\underset{x \to a}{\text{Lt}} \left[\frac{f_1(x)}{f_2(x)}\right] = \frac{\underset{x \to a}{\text{Lt}} f_1(x)}{\underset{x \to a}{\text{Lt}} f_2(x)}$ provided $\underset{x \to a}{\text{Lt}} f_2(x) \neq 0$

Continuity. A function $f(x)$ defined for $x = a$ is said to be continuous at $x = a$, if

(*i*) the value of $f(x)$ at $x = a$ is a definite number

(*ii*) the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Indeterminate Forms

1. Form $\frac{0}{0}$. If $f(a) = \phi(a) = 0$, then

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \quad \text{[L' Hospital's rule]}$$

In general

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \frac{f^n(a)}{\phi^n(a)} = \underset{x \to a}{\text{Lt}} \frac{f^n(x)}{\phi^n(x)}$$

2. Form $\frac{\infty}{\infty}$. If $f(a) = \phi(a) = \infty$, then

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} = \underset{x \to a}{\text{Lt}} \frac{f^n(x)}{\phi^n(n)}$$

3. Forms reducible to $\frac{0}{0}$ form.

(*a*) Form $0 \times \infty$. If $\underset{x \to 0}{\text{Lt}} f(x) = 0$ and $\underset{x \to \infty}{\text{Lt}} \phi(x) = \infty$, then

$$f(x).\ \phi(x) = \frac{f(x)}{[1/\phi(x)]} \text{ to take the form } \frac{0}{0}$$

or

$$= \frac{\phi(x)}{[1/f(x)]} \text{ to take the form } \frac{\infty}{\infty}$$

(*b*) Form $\infty - \infty$. If $\underset{x \to a}{\text{Lt}} f(x) = \infty$ and $\underset{x \to a}{\text{Lt}} \phi(x) = \infty$, then

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \left[\frac{1}{f(x)\,\phi(x)}\right]$$

(*c*) Form 0^o, 1^∞, ∞^o. If $y = \underset{x \to a}{\text{Lt}} [f(x)]^{\phi(x)}$, then

$\log_e y = \underset{x \to a}{\text{Lt}} \phi(x) \log_e f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in (*a*) above.

Example 1.5 *Evaluate the following limits*:

(*a*) $\underset{x \to 1}{Lt} \dfrac{x^x - x}{x - 1 - \log x}$ **(*b*)** $\underset{x \to 0}{Lt} \dfrac{\log x}{\cot x}$

(*c*) $\underset{x \to 0}{Lt} \left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$ **(*d*)** $\underset{x \to \frac{\pi}{2}}{Lt} (\sin x)^{\tan x}$

Solution. (*a*) $\underset{x \to 1}{\text{Lt}} \dfrac{x^x - x}{x - 1 - \log x} = \dfrac{1-1}{1-1-0} = \dfrac{0}{0}$ form

Using *L'* Hospital's rule $\underset{x \to 1}{\text{Lt}} \dfrac{\frac{d}{dx}(x^x) - 1}{1 - 0 - \frac{1}{x}}$

Let

$$y = x^x$$

$$\log y = x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + 1 \times \log x = 1 + \log x$$

$$\frac{dy}{dx} = x^x(1 + \log x)$$

$$\underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{1 - \frac{1}{x}} = \frac{1 - 1}{1 - 1} = \frac{0}{0} \text{ form}$$

$$\underset{x \to 1}{\text{Lt}} \frac{\frac{d}{dx}(x^x) \cdot (1 + \log x) + x^x \cdot \frac{1}{x} - 0}{\frac{1}{x^2}}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x (1 + \log x)^2 + x^x \cdot \frac{1}{x}}{\frac{1}{x^2}} = \frac{1 + 1}{1} = 2$$

(*b*)

$$\underset{x \to 0}{\text{Lt}} \frac{\log x}{\cot x} = \frac{\infty}{\infty} \text{ form}$$

$$\underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x}}{-\text{cosec}^2 x} = -\underset{x \to 0}{\text{Lt}} \frac{\sin^2 x}{x} = \frac{0}{0} \text{ form}$$

$$-\underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0$$

(*c*)

$$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \infty - \infty \text{ form}$$

$$\underset{x \to 0}{\text{Lt}} \left(\frac{x - \sin x}{x \sin x} \right) = \frac{0}{0} \text{ form}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{x \cos x + \sin x} = \frac{0}{0} \text{ form}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{-x \sin x + \cos x + \cos x} = \frac{0}{0 + 1 + 1} = \frac{0}{2} = 0$$

(*d*)

$$\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sin x)^{\tan x} = 1^{\infty} \text{ form}$$

Let

$$y = \underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sin x)^{\tan x}$$

$$\log_e y = \underset{x \to \frac{\pi}{2}}{\text{Lt}} \tan x \log_e \sin x$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\log_e \sin x}{\cot x} = \frac{0}{0} \text{ form}$$

$$= \underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\left(\frac{1}{\sin x}\right) \cos x}{-\text{cosec}^2 x} = -\underset{x \to \frac{\pi}{2}}{\text{Lt}} \sin x \cos x = 0$$

$$y = e^0 = 1$$

Rolle's Theorem

If **(*i*)** $f(x)$ is continuous in the closed interval $[a, b]$,

(*ii*) $f'(x)$ exists for every value of x in the open interval (a, b), and

(*iii*) $f(a) = f(b)$,

then there is at least one value of x in (a, b) such that $f'(c) = 0$

Lagrange's Mean-value Theorem

(*a*) If **(*i*)** $f(x)$ is continuous in the closed interval (a, b), and

(*ii*) $f'(x)$ exists in the open interval (a, b), then these is at least one value c of x in (a, b), such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

(*b*) If **(*i*)** $f(x)$ is continuous in the closed interval $[a, a + h]$, and

(*ii*) $f'(x)$ exists in the open interval $(a, a + h)$, then there is at least one number $\theta(0 < \theta < 1)$, such that

$$f(a + h) = f(a) + hf'(a + \theta h)$$

Taylor's Theorem

If **(*i*)** $f(x)$ and its first $(n - 1)$ derivatives be continuous in $[a, a + h]$, and

(*ii*) $f^n(x)$ exists for every value of x in $(a, a + h)$, then there is at least one number θ $(0 < \theta < 1)$, such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a + \theta h)$$

1.3.2 Series

Maclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... \infty$$

Well-known Series

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...$$

$$\sin h\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + ...$$

$$\cos h\theta = 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + ...$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5 + ...$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} ...$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

$$\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\right)$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

Taylor's Series

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \infty$$

1.3.3 Partial Differentiation

Functions of two or more variables

Limits. The function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$.

$$\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} f(x, y) = l$$

Continuity. A function $f(x, y)$ is said to be continuous at the point (a, b) if

$$\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} f(x, y) \text{ exists and } = f(a, b)$$

If $\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} f(x, y) = l$ and $\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} g(x, y) = m$, then

(*i*) $\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} [f(x, y) \pm g(x, y)] = l \pm m$

(*ii*) $\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} [f(x, y) \cdot g(x, y)] = l \cdot m$

(*iii*) $\operatorname*{Lt}_{\substack{x \to a \\ y \to b}} [f(x, y) / g(x, y)] = l/m$

Example 1.6 *If $f(x) = x(x - 1)(x - 2)$. Determine c lying between a and b if $a = 0$ and $b = \frac{1}{2}$, using mean value theorem.*

Solution.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(a) = f(0) = 0$$

$$f(b) = f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right) = \frac{3}{8}$$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

$$\therefore \quad 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{4}$$

$$3c^2 - 6c + \frac{5}{4} = 0$$

or

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{576 - 240}}{24} = 1 \pm 0.764 = 1.764,\ 0.236$$

Example 1.7 *Evaluate the following limit:*

$$\operatorname*{Lt}_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$$

Solution. $$\underset{y \to 0}{\text{Lt}} \left[\underset{x \to 0}{\text{Lt}} \frac{xy}{x^2 + y^2} \right]$$

$$= \underset{y \to 0}{\text{Lt}} \frac{y}{y^2} = \underset{y \to 0}{\text{Lt}} \frac{1}{y} = \infty$$

∴ Limit does not exist.

1.4 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y. The derivative of z *w.r.t.* x, treating y as constant, is called the partial derivative of z *w.r.t.* x. It is denoted by $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$, $f_x(x, y)$, $D_x(f)$.

Higher partial derivatives can be obtained by further differentiation.

$$\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{ and so on.}$$

Total Derivative

(*i*) If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

(*ii*) If $f(x, y) = c$ be an implicit relation between x and y, then

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0, \text{ giving}$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

(*iii*) Similarly, if $u = f(x, y, z)$ where x, y, z are functions of t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

(*iv*) If $T = f(p,v)$, then

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} . dv$$

(*v*) If $u = f(x, y)$, where $x = \phi(s, t)$ and $y = \psi(s, t)$, then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

and $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Taylor's Theorem for Functions of Two Variables

$$f(x, y) = f(a, b) + [(x - a) f_x(a, b) + (y - b) f_y(a, b)]$$
$$+ \frac{1}{2!} [(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \ldots$$

Total Differential

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

1.4.1 Maxima and Minima of Two Variables

A function $f(x, y)$ is said to have a maximum or minimum at $x = a$, $y = b$, accordingly as

$$f(a + h, b + k) \quad < \text{or} > \quad f(a, b)$$

For all positive or negative small values of h and k.

(*i*) The necessary conditions for $f(x, y)$ to have a maximum or a minimum at (a, b) are that

$$f_x(a, b) = 0, \quad f_y(a, b) = 0$$

(*ii*) Sufficient condition for maxima and minima.

Let $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$, $t = f_{yy}(a, b)$

(*a*) If $(rt - s^2) > 0$, then $f(x, y)$ has a maxima or a minima at (a, b) accordingly as $r <$ or > 0.

(*b*) If $(rt - s^2) < 0$, then $f(x, y)$ will have neither a maximum nor a minimum at (a, b), *i.e.,* it is a saddle point.

(*c*) If $(rt - s^2) = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Example 1.8 *Determine the maxima or minima of* $f(x, y) = x^3y^2\ (1 - x - y)$

Solution.

$$\begin{aligned} f_x &= 3x^2y^2 - 4x^3y^2 - 3x^2y^3 \\ f_y &= 2x^3y - 2x^4y - 3x^3y^2 \\ r &= f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y) \\ s &= f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y\ (6 - 8x - 9y) \\ t &= f_{yy} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1 - x - 3y) \end{aligned}$$

For $f_x = 0$, $f_y = 0$, we have

$$\begin{aligned} x^2y^2\ (3 - 4x - 3y) &= 0 \\ x^3y\ (2 - 2x - 3y) &= 0 \end{aligned}$$

Solving these two equations, we get

$$(0, 0), \ \left(\frac{1}{2}, \frac{1}{3}\right)$$

$$\text{At } \left(\frac{1}{2}, \frac{1}{3}\right), rt - s^2 = 6 \times \frac{1}{2} \times \frac{1}{9}\left(1 - 2 \times \frac{1}{2} - \frac{1}{3}\right) \times 2 \times \frac{1}{8}\left(1 - \frac{1}{2} - 1\right) - \left[\frac{1}{4} \times \frac{1}{3}(6 - 4 - 3)\right]^2$$

$$= +\frac{1}{12} \times \frac{1}{3} \times \frac{1}{2} - \frac{1}{144} = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$$

Also $$r = 6 \times \frac{1}{2} \times \frac{1}{9}\left(1 - 1 - \frac{1}{3}\right) = -\frac{1}{9} < 0$$

Hence, $f(x, y)$ has a maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$

Maximum value, $$f(x, y) = \frac{1}{8} \times \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

At $(0, 0)$, $rt - s^2 = 0$ and therefore, further investigation is required. For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$, which is +ve for $x = 0.1$ and –ve for $x = -0.1$, *i.e.,* in the neighbourhood of $(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

1.5 INTEGRATION

1.5.1 Definite Integrals

$$\int_a^b f(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

where, $f(x)$ is the integral of $f(x)$.

Properties:

(*i*) $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

(*ii*) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(*iii*) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(*iv*) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(*v*) $$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even function of } x$$
$$= 0 \text{ if } f(x) \text{ is odd function of } x.$$

(*vi*) $$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)$$
$$= 0, \text{ if } f(2a-x) = -f(x)$$

(*vii*) $$\int_a^b f_1(x)\,f_2(x)\,dx = |\,f_1(x)\,|_a^b \int_a^b f_2(x)\,dx - \int_a^b f'_1(x)\,dx \int_a^b f_2(x)\,dx$$

Improper Integral

The definite integral $\int_0^a f(x)\,dx$ is called *improper integral*, if

(*i*) The range of integration is infinite and the integrand is bounded.
(*ii*) The range of integration is definite and the integrand is unbounded.
(***iii***) Neither the range of integration is finite nor integrand is bounded.

1.5.2 Multiple Integrals

Double Integral

The integral $\iint_A f(x, y)\,dx\,dy$ is called the double integral of (x, y) over the region A.

$$I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\,dy\,dx = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y)\,dx\,dy$$

The integration is carried from inner to the outer variable.

Example 1.9 *Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.*

Solution.
$$y = \frac{x^2}{4a}$$
$$\frac{x^4}{16a^2} = 4ax$$

$$x^3 = 64a^3$$
$$x = 4a$$
$$y = \frac{16a^2}{4a} = 4a$$

Therefore, the parabolas intersect at $(4a, 4a)$

$$\text{Area} = \int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\, dx$$
$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a} \right) dx$$
$$= \left| 2\sqrt{a}.\, \frac{x^{3/2}}{(3/2)} - \frac{x^3}{12a} \right|_0^{4a}$$
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

1.5.3 Triple Integral

The integral $\iiint_V f(x, y, z)\ dx\ dy\ dz$ is called the triple integral over the volume V.

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,\, y)}^{z_2(x,\, y)} f(x, y, z)\, dz\, dy\, dx$$

1.6 INFINITE SERIES

If $u_1, u_2, u_3, ..., u_n, ...$ be an infinite sequence of real numbers, then $\sum_{i=1}^{\infty} u_i$ is called an *infinite series*. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by S_n.

(*i*) If $S_n \to$ finite limit as $n \to \infty$, the series Σu_n is said to be convergent.

(*ii*) If $S_n \to \pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

(*iii*) If S_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be oscillatory or non-convergent.

Example 1.10 *Show that the series* $1 + r + r^2 + r^3 + ... \infty$ **(*i*)** *converges if* $|r| < 1$, **(*ii*)** *diverges if* $r \geq 1$, *and* **(*iii*)** *oscillates if* $r \leq -1$.

Solution. Let $S_n = 1 + r + r^2 + ... + r^{n-1}$

(*i*) When $|r| < 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n = 0$

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r}$$
$$\underset{n \to \infty}{\text{Lt}}\ S_n = \frac{1}{1 - r}, \text{ which is a finite limit.}$$

$\therefore$ The series is convergent.

(*ii*) When $r > 1$, $\underset{n \to \infty}{\text{Lt}}\ r^n \to \infty$

$$S_n = \frac{r^n - 1}{r - 1} = \frac{r^n}{r - 1} - \frac{1}{r - 1}$$

$$\underset{n \to \infty}{\text{Lt}} \; s_n \to \infty$$

∴ The series is divergent.

when $r = 1$, then $S_n = n$

$$\underset{n \to \infty}{\text{Lt}} \; s_n \to \infty$$

∴ The series is divergent.

(*iii*) When $r = -1$, then the series becomes $1 - 1 + 1 - 1...$, which is an oscillatory series. When $r < -1$, let $r = -\rho$ so that $\rho > 1$. Then $r^n = (-1)^n \rho^n$

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1 - (-1)^n \rho^n}{1 + \rho}. \; \underset{n \to \infty}{\text{Lt}} \; \rho^n \to \infty$$

∴ $\underset{n \to \infty}{\text{Lt}} \; S_n \to -\infty$ or $+\infty$ accordingly as n is even or odd. Hence, the series oscillates.

1.6.1 Series Tests

Geometric Series Test

The geometric series $a + ar + ar^2 + ...$ is

(*i*) Convergent if $|r| < 1$

(*ii*) Divergent if $r \geq 1$

(*iii*) Oscillatory if $r \leq -1$

Hyperharmonic or p-series Test

The infinite series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ...$ is

(*i*) Convergent if $p > 1$

(*ii*) Divergent if $p \leq 1$

Gauss's Test

If Σu_n is a series of positive terms, and if the ratio $\frac{u_n}{u_{n+1}}$ is expressed in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\mu}{n^2} + \frac{\alpha}{n^3} + ..., \text{ then the series is}$$

(*i*) Convergent if $\lambda > 1$

(*ii*) Divergent if $\lambda \leq 1$

De-Morgan and Bertrand's Test

If Σu_n is a positive term series, and if

$$\underset{n \to \infty}{\text{Lt}} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = \lambda, \text{ then the series is}$$

(*i*) Convergent for $\lambda > 1$

(*ii*) Divergent for $\lambda < 1$

(*iii*) The test fails for $\lambda = 1$

General Properties of Series

(*i*) The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms.

(*ii*) If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative.

(*iii*) The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.

(*iv*) The necessary condition for convergence of a positive terms series is :

$$\underset{n \to \infty}{\text{Lt}} u_n = 0$$

If $\underset{n \to \infty}{\text{Lt}} u_n \neq 0$, the series Σu_n diverges.

Comparison Tests

(*i*) If two positive term series Σu_n and Σv_n be such that (*a*) Σv_n converges, (*b*) $u_n \leq v_n$ for all values of n, then Σu_n also converges.

(*ii*) If two positive term series Σu_n and Σv_n be such that (*a*) Σv_n diverges, (*b*) $u_n \geq v_n$ for all values of n, then Σu_n also diverges.

(*iii*) If two positive term series Σu_n and v_n be such that $\underset{n \to \infty}{\text{Lt}} \dfrac{u_n}{v_n}$ = finite quantity ($\neq 0$), then Σu_n and Σv_n converge or diverge together.

Integral Test

A positive term series $f(1) + f(2) + \ldots + f(n) + \ldots$, where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)\, dx + f(1)$ is finite or infinite.

1.6.2 Comparison of Ratios

If Σu_n and Σv_n be two positive term series, then Σu_n converges if **(*i*)** Σv_n converges, and **(*ii*)** from and after some particular term,

$$\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$$

Similarly, Σu_n diverges, if (*i*) Σv_n diverges, and (*ii*) from and after a particular term $\dfrac{u_n}{u_{n+1}} > \dfrac{v_{n+1}}{v_n}$

D'Alembert's Ratio Test

In a positive term series Σu_n, if

$$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \lambda,$$ then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Ratio test fails when $\lambda = 1$.

Further Tests for Convergence

In the positive term series Σu_n, if $\underset{n \to \infty}{\text{Lt}} \dfrac{u_n}{u_{n+1}} = k$, then

(*i*) the series converges for $k > 1$,

(*ii*) diverges for $k < 1$, and

(*iii*) the test fails for $k = 1$.

When the ratio test fails, we apply the following tests :

1. **Raabe's Test.** In the positive term series Σu_n, if $\underset{n \to \infty}{\text{Lt}}\, n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$, and diverges for $k < 1$, but the test fails for $k = 1$.

2. **Logarithmic Test.** In the positive term series Σu_n, if $\underset{n \to \infty}{\text{Lt}}\left(n \log \dfrac{u_n}{u_{n+1}}\right) = k$, then the series converges for $k > 1$, and diverges for $k < 1$, but the test fails for $k = 1$.

Cauchy's Root Test

In a positive series Σu_n, if $\underset{n \to \infty}{\text{Lt}}\, (u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$, and diverges for $\lambda > 1$.

Alternating Series

A series in which the terms are alternately positive or negative is called an alternating series.

Leibnitz's Rule

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ converges, if (*i*) each term is numerically less than its preceding term, and (*ii*) $\underset{n \to \infty}{\text{Lt}}\, u_n = 0$.

If $\underset{n \to \infty}{\text{Lt}}\, u_n \neq 0$, the given series is oscillatory.

Series of Positive or Negative Terms

(*i*) If the series of arbitrary terms $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ be such that the series $|u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots$ is convergent, then the series Σu_n is said to be absolutely convergent.

(*ii*) If $\Sigma\, |u_n|$ is divergent but Σu_n is convergent, then Σu_n is said to be conditionally convergent.

Power Series

A series of the form $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ where the *a's* are independent of *x*, is called a power series in *x*. Such a series may converge for some or all values of *x*.

In the power series, $u_n = a_n x^n$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}} \frac{a_{n+1}\, x^{n+1}}{a_n x^n} = \underset{x \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) x$$

$$\underset{n \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) = l, \text{ then the series}$$

(*i*) Converges when $|x| < \dfrac{1}{l}$, and

(*ii*) Diverges for other values.

Thus, the power series converges within the interval $-\dfrac{1}{l} < x < \dfrac{1}{l}$ and diverges for values of *x* outside this interval.

1.6.3 Convergence of Exponential Series

The series $1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... \infty$ is convergent for all values of x.

Because $$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}} \left[\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!}\right] = \underset{n \to \infty}{\text{Lt}} \frac{x}{n} = 0$$

Convergence of Logarithmic Series

The series $x - \frac{x^2}{2} + \frac{x^3}{3} ... + (-1)^n \frac{x^n}{n} + ... \infty$ is convergent for $-1 < x \le 1$

Here, $$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}} \frac{(-1)^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n}$$

$$= -x \underset{n \to \infty}{\text{Lt}} \frac{n}{n+1} = -x \underset{n \to \infty}{\text{Lt}} \left\{\frac{1}{1 + \frac{1}{n}}\right\} = -x$$

Hence, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

when $x = 1$, the series is convergent.

when $x = -1$, the series is divergent.

Hence, the series converges for $-1 < x \le 1$

Convergence of Binomial series

The series $1 + nx + \frac{n(n-1)}{2!} x^2 + ... + \frac{n(n-1)...(n-r+1)}{r!} x^r + ... \infty$ converges for $|x| < 1$.

Uniform Convergence

The series $\Sigma u_n(x)$ is said to be uniformly convergent in the interval (a, b), if for a given $\varepsilon > 0$, a number N can be found independent of x, such that for every x in the interval (a, b),

$$|S(x) - S_n(x)| < \varepsilon \text{ for all } n > N$$

Weierstrass's M-Test

A series $\Sigma u_n(x)$ is uniformly convergent in an interval (a, b), if there exists a convergent series ΣM_n of positive constants such that $|u_n(x)| \le M_n$ for all values of x in (a, b).

1.6.4 Fourier Series

The Fourier series for the function $f(x)$ in the interval $\alpha \angle x \angle \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where, $$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha + 2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha + 2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha + 2\pi} f(x) \sin nx\, dx$$

The conditions on $f(x)$ to be expanded as Fourier series are :

(*i*) $f(x)$ is periodic, single-valued and finite.

(*ii*) $f(x)$ has a finite number of discontinuities in any one period.

(*iii*) $f(x)$ has at the most a finite number of maxima and minima.

Change of Interval

For the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where,

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Even and Odd Functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$.

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$.

A periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx\ ,\quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx\ ,\quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

(*i*) when $f(x)$ is an even function

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = 0$$

(*ii*) when $f(x)$ is an odd function

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

Half-Range Series

(*i*) Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}\,dx$$

where, $$b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

(*ii*) Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

where, $$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx$$

1.7 VECTOR CALCULUS

Divergence of a Vector

The divergence of a continuously differentiable vector point function F is denoted by div. F and is defined by the equation

$$\text{div. } F = \nabla\cdot F = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$$

where i, j, k are unit vectors.

If $$F = F_x i + F_y j + F_z k, \text{ then}$$

$$\nabla\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(F_x i + F_y j + F_z k)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

∇f is gradient of the scalar point function f and is written as grad f.

$$\text{grad } f = \nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

The grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along its normal.

Curl of Vector

The curl of a continuously differentiable vector point function F is defined by the equation

$$\text{curl } F = \nabla\times F = i\times\frac{\partial F}{\partial x} + j\times\frac{\partial F}{\partial y} + k\times\frac{\partial F}{\partial z}$$

If $$F = F_x i + F_y j + F_z k, \text{ then}$$

$$\text{Curl } F = \nabla\times F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(F_x i + F_y j + F_z k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = i\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + j\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + k\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

Physical Interpretation

(*i*) Div V gives the rate at which fluid is originating at a point per unit volume. If div $V = 0$ everywhere then such a point function is called a *solenoid vector function*.

(*ii*) The curl of any vector point function gives the measure of the angular velocity at any point of the vector field. Any motion in which the curl of the velocity vector is zero is said to be irrotational, otherwise rotational.

Some important operations :

$$\begin{aligned}
\text{div grad } f &= \nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\text{curl grad } f &= \nabla \times \nabla f = 0 \\
\text{div curl } F &= \nabla \cdot \nabla \times f = 0 \\
\text{curl curl } F &= \text{grad div } F - \nabla^2 F = \nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F \\
\text{grad div } F &= \text{curl curl } F + \nabla^2 F = \nabla(\nabla \cdot F) = \nabla \times (\nabla \times F) + \nabla^2 F
\end{aligned}$$

Integration of Vectors

If two vector functions $F(t)$ and $G(t)$ be such that

$$\frac{dG(t)}{dt} = F(t), \text{ then}$$

$$\int_a^b F(t)\, dt = [G(t) + C]_a^b = G(b) - G(a)$$

Line Integral

Consider a continuous vector function $F(R)$ which is defined at each point of curve C in space. The tangential line integral of $F(R)$ along C is written as

$\int_c F(R)\, dR$ or $\int_c F \cdot \frac{dR}{dt} dt$ or $\oint_c F \cdot \frac{dR}{dt} dt$, when the path of integration is a closed curve.

Surface Integral

Consider a continuous function $F(R)$ and a surface S. The normal surface integral of $F(R)$ over S is denoted by

$\int_s F \cdot ds$ or $\int_s F \cdot \bar{n}\, ds$ where $\bar{n}$ is a unit outward normal to S.

Green's Theorem in the Plane

If $\phi\,(x, y)$, $\psi(x, y)$, ϕ_y and ψ_x be continued in a region E of the xy-plane bounded by a closed curve C, then

$$\int_c (\phi\, dx + \psi\, dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy$$

This theorem connects a line integral around a closed curve into a double integral.

Stoke's Theorem

If S be an open surface bounded by a closed curve C and $F = F_x i + F_y j + F_z k$ be any continuously differentiable vector point function, then

$$\int_c F \cdot dR = \int_s curl\ F \cdot \hat{n}\, ds$$

where $\hat{n} = \cos \alpha\, i + \cos \beta\, j + \cos \gamma\, k$ is a unit external normal at any point of S.

Volume Integral

Consider a continuous vector function $F(R)$ and volume V enclosing the region E. The volume integral of $F(R)$ over E is written as $\int_E F\,dv$.

If $F(R) = F_x(x, y, z)\, i + F_y\,(x, y, z)\, j + F_z\,(x, y, z)\, k$, so that $dv = dx\ dy\ dz$, then

$$\int_E F\,dv = i \iiint_E F_x dx\ dy\ dz + j \iiint_E F_y\ dx\ dy\ dz + k \iiint_E F_z dx\ dy\ dz$$

Gauss Divergence Theorem

If F is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S F.\hat{n}\,ds = \int_E div\ F\ dv$$

where $\hat{n}$ is the unit external normal vector

If $F(R) = F_x\,(x, y, z)\, i + F_y\,(x, y, z)\, j + F_z\,(x, y, z)\, k$, then

$$\iint_S (F_x\ dy\,dz + F_y\ dz\ dx + F_z\ dx\,dy) = \iiint_E \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dxdydz$$

Green's Theorem

If ϕ and ψ are scalar point functions possessing continuous derivatives of first and second orders, then

$$\int_E (\phi\,\nabla^2\,\psi - \psi\,\nabla^2\,\phi)\ dv = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Harmonic Function

A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.

Greens' Reciprocal Theorem

If ϕ and ψ be both harmonic functions in E, then

$$\int_S \phi\frac{\partial\psi}{\partial n}\,ds = \int_S \psi\frac{\partial\phi}{\partial n}\,ds$$

1.8 ORDINARY DIFFERENTIAL EQUATIONS

In an ordinary differential equation, the differential coefficients have reference to a single independent variable.

Equations of First Order and First Degree.

1. **Variables Separable**

$$f(y)\ dy = \phi(x)\ dx$$

its solution is, $\int f(y)\, dy = \int \phi(x)\, dx + c$

2. Homogeneous equations

$$\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$$

where, $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree.

To obtain the solution, ***(i)*** put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(ii) Separate the variables v and x, and integrate.

3. Equations reducible to homogeneous form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

(*a*) when $\frac{a}{a'} \neq \frac{b}{b'}$

Put $x = X + h$, $y = Y + k$, so that $dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')}$$

choose h, k so that the above equation becomes homogeneous.

Put $ah + bk + c = a'h + b'k + c' = 0$, so that

$$\frac{h}{bc' - b'c} = \frac{h}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - ba'}, \; k = \frac{ca' - c'a}{ab' - ba'}$$

when $ab' - ba' \neq 0$,

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y},$$

which is homogeneous and can be solved by putting $Y = vX$

(*b*) when $\frac{a}{a'} = \frac{b}{b'}$, *i.e.*, $ab' - ba' = 0$

Let $$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$$

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c'}$$

Put $ax + by = t$ so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

Now put $t = ax + by$ to get the solution

Linear Equations

$$\frac{dy}{dx} + Py = Q$$

where, P, Q are functions of x.

Integrating Factor (I.F.) = $\int\limits_{e} Pdx$

The solution is $Y(\text{I.F.}) = \int Q.\,(I.F.)\,dx + c$

Equations reducible to the Linear Form (Bernoulli's equation)

$$\frac{dy}{dx} + Py = Qy^n$$

Divide both sides by y^n, so that

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$, to obtain

$$\frac{1}{1-n}\frac{dz}{dx} + PZ = Q$$

or $$\frac{dz}{dx} + P\,(1-n)\,z = Q\,(1-n)$$

which can be solved.

Exact Differential Equations

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Its solution is,

$$\int\limits_{(y\ const.)} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

Integrating Factor

$$d\left(\frac{x}{y}\right) = \frac{xdy - ydx}{x^2} \text{ or } \frac{ydx - xdy}{y^2}$$

$$d(xy) = xdy + ydx$$

$$d\left(\frac{x^2}{y}\right) = \frac{2yx\,dx - x\,dy}{y^2}$$

$$d\left(\frac{y^2}{x^2}\right) = \frac{2yx^2\,dy - 2xy^2\,dx}{4}$$

$$= \frac{-2xy^2\,dx}{x}$$

$$d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d[\tan^{-1}(x/y)] = \frac{xdy - ydx}{x^2 + y^2}$$

$$d\left[\frac{1}{2}\log(x^2 + y^2)\right] = \frac{xdx + ydy}{x^2 + y^2}$$

$$d\left(-\frac{1}{xy}\right) = \frac{xdy + ydx}{x^2\,y^2}$$

$$d\log\left(\frac{x}{y}\right) = \frac{ydx - xdy}{xy}$$

$$d\log\left(\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$$

$$d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$$

$Mdx + Ndy = 0$, $M = yf_1(xy)$, $N = xf_2(xy)$, If $= \dfrac{1}{Mx - Ny}$, $Mx - Ny \neq 0$.

$Mdx + Ndy = 0$, if $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is a function of x alone, then $I.F. = e^{\int f(x)\,dx}$

$Mdx + Ndy = 0$, if $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of x alone, then $I.F. = e^{\int f(y)\,dy}$

If the equation $Mdx + Ndy = 0$ is homogeneous then $\dfrac{1}{Mx + Ny}$ is an I.F. provided $Mx + Ny \neq 0$.

Particular Integral (P.I.)

$$\text{P.I.} = \frac{1}{D^n + k_1 D^{n-1} + \ldots + k_n} X$$

(*i*) $X = e^{ax}$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{if } f(a) \neq 0$$

$$= x \frac{1}{f'(a)} e^{ax} \quad \text{if } f(a) = 0$$

$$= x^2 \frac{1}{f''(a)} e^{ax} \quad \text{if } f(a) = 0$$

and so on.

(*ii*) $X = \sin(ax + b)$ or $\cos(ax + b)$

$$\frac{1}{f(D^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (ax + b) = \frac{1}{f(-a^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (ax + b), \text{ if } f(-a^2) \neq 0$$

$$= x \frac{1}{f'(-a^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (ax+b), \text{ if } f'(-a^2) \neq 0$$

$$= x^2 \frac{1}{f''(-a^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (ax+b), \text{ if } f''(-a^2) \neq 0$$

(iii) $X = x^m$

$$\frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term.

(iv) $X = e^{ax} V$, V being a function of x.

$$\frac{1}{f(D)} \left(e^{ax} V\right) = e^{ax} \frac{1}{f(D+a)} V$$

(v) $X = xV$

$$\frac{1}{f(D)} (xV) = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2}$$

(vi) X is any function of x.

$$\frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X.

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax} \, dx$$

1.9 LINEAR DIFFERENTIAL EQUATIONS

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = X$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant coefficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = X$$

where, k_i, $i = 1$ to n are constants.

Denoting $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, *etc.*, so that $\frac{dy}{dx} = Dy$, $\frac{d^2 y}{dx^2} = D^2 y$, *etc.*, we have

$$(D^n + k_1 D^{n-1} + \ldots + k_n) y = X$$

Complementary Function (C.F.)

So solve the equation $(D^n + k_1 D^{n-1} + \ldots + k_n) y = 0$

Its symbolic coefficient equated to zero, *i.e.*,

$$D^n + k_1 D^{n-1} + ... + k_n = 0$$

is called the auxiliary equation (A.E.)

Let $m_1, m_2, ..., m_n$ be its roots

(*i*) If all the roots be real and different, then

$(D - m_1)(D - m_2) ... (D - m_n)y = 0$

Its solution is , $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + ... + c_n e^{m_n x}$

(*ii*) If two roots are equal (*i.e.*, $m_1 = m_2$), then

$$y = (c_1 + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

If three roots are equal (*i.e.*, $m_1 = m_2 = m_3$), then

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + ... + c_n e^{m_n x}$$

(*iii*) If one pair of roots is imaginary, *i.e.*, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

where, $C_1 = c_1 + c_2,\ C_2 = i(c_1 - c_2)$

Complete Solution (C.S.) = C.F. + P.I.

Method of Variation of Parameters

This method is applicable to equations of the form

$$y'' + py' + qy = X$$

where, p, q and X are functions of X.

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

where, y_1 and y_2 are the solutions of $y'' + py' + qy = 0$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called the Wronskian of y_1, y_2.

Method of undetermined coefficients

To find P.I. of $f(D)\, y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation.

The trial solution to be assumed in each case, depends on the form of X.

1.10 PARTIAL DIFFERENTIAL EQUATIONS

Linear Equations of the First Order

$$Pp + Qq = R \text{ (Lagrange's linear equation)}$$

where P, Q and R are functions of x, y, z. When P, Q, R are independent of z, it is known as linear equation.

To obtain the solution, **(*i*)** form the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(*ii*) solve these simultaneous equations giving $u = a$ and $v = b$ as its solutions.

(*iii*) write the complete solution as $\phi(u, v) = 0$ or $u = f(v)$

Laplace Transforms

Let $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written at $\bar{f}(s)$,

i.e., $$L\{f(t)\} = \bar{f}(s)$$

Inverse Laplace transform of $\bar{f}(s)$

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

Transforms of Elementary Functions

$$L(1) = \frac{1}{s} \qquad (s > 0)$$

$$L(t^n) = \frac{n!}{s^{n+1}}, \qquad \text{where } n = 0, 1, 2...$$

$$L(e^{at}) = \frac{1}{s-a} \qquad (s > a)$$

$$L(\sin at) = \frac{a}{s^2 + a^2} \qquad (s > 0)$$

$$L(\cos at) = \frac{s}{s^2 + a^2} \qquad (s > 0)$$

$$L(\sinh at) = \frac{a}{s^2 - a^2} \qquad (s > |a|)$$

$$L(\cosh at) = \frac{s}{s^2 - a^2} \qquad (s > |a|)$$

Properties of Laplace Transforms

1. Linearity property. If a, b, c be any constants and f, g, h any functions of t, then $L[af(t) + bg(t) - ch(t)] = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$
2. First shifting property. If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$

Useful Results

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$L(e^{at}\sin bt) = \frac{b}{(s-a)^2 + b^2}$$